Adaptive Quadrature:

Goal: Evaluate $\int_a^b f(x)\,dx$ accurate to within an error bound.

Questions:

1. How many subdivisions should we use?
2. Which method? (eg Simpson's $\frac{3}{8}$, Trap)
3. Should different parts of $[a,b]$ be subdivided more than others?

Idea: We will use our computations to estimate the error in different parts of the graph.

Consider a small interval $[a, b]$, let's use trapezoid rule:

$$\int_a^b f(x)\,dx \approx \mathrm{TRAP}(\underset{\substack{\uparrow \\ [a,b]}}{1}) = \frac{(b-a)}{2}\left(f(a)+f(b)\right)$$

Error

$$E(1) = \int_a^b f(x)\,dx - \mathrm{TRAP}(1) = -\frac{(b-a)^3}{12\,n^2} f''(\xi_1)$$

$$= -\frac{(b-a)^3}{12} f''(\xi_1) \quad \text{(since } n=1\text{)}$$

$$E(2) = \int_a^b f(x)\,dx - TRAP(2) = -\frac{(b-a)^3}{12n^2} f''(\xi_2)$$
$$= -\frac{(b-a)^3}{12 \cdot 4} f''(\xi_2)$$

If $[a,b]$ is very small, $f''(\xi_1) \approx f''(\xi_2)$
as long as f'' is continuous.

$$\Rightarrow \quad E(2) \approx \frac{1}{4} E(1) \Rightarrow E(1) = 4E(2)$$

$$\Rightarrow \quad \int_a^b f(x)\,dx - TRAP(1) \approx 4E(2)$$
$$\int_a^b f(x)\,dx - TRAP(2) = E(2)$$

subtract
$$TRAP(2) - TRAP(1) \approx 3E(2)$$
$$\Rightarrow \quad E(2) \approx \frac{TRAP(2) - TRAP(1)}{3}.$$

Bonus: We can estimate the error $E(2)$ from going from $TRAP(1) \to TRAP(2)$ for one particular interval, just from the values of trap(1) & TRAP(2).

(This same technique would work for any of the integration estimation methods)

An adapted trapezoid rule algorithm:

$\int_a^b f(x)\,dx = ?$ want this accurate within an error $|\text{error}| < E_0$.

① Calculate TRAP(1) & TRAP(2).

② Estimate $E(2) \simeq \dfrac{TRAP(2) - TRAP(1)}{3}$.

If $|E(2)| < E_0$, we're done.

③ If not, split the interval in two and the error in 2

List: $\left\{\left[a, \frac{a+b}{2}\right], \frac{E_0}{2}\right\}, \left\{\left[\frac{a+b}{2}, b\right], \frac{E_0}{2}\right\}$

④ Go back to step 1 for each of these.

TRAP(1)
TRAP(2)
$E(2) \simeq$
Is $|E(2)| < \frac{E_0}{2}$?
Yes → done

TRAP(1)
TRAP(2)
$E(2) \simeq$.
Is $|E(2)| < \frac{E_0}{2}$?
Yes → done

Typically, computer algorithms will be more complicated: eg.

① Randomly divide interval into 20 pieces. Divide E_0 proportionately.

② Then do adapted method.

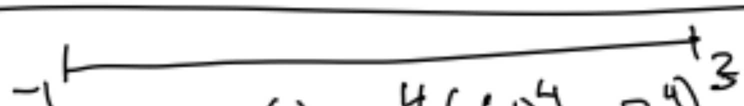
Example to compute an integral using the adapted trapezoid method.

We wish to calculate

$$\int_{-1}^{3} x^4 \, dx$$ within an abs. error of .5

$TRAP(1) = \frac{h}{2}(y_0 + y_1)$

$TRAP(2) = \frac{h}{2}(y_0 + 2y_1 + y_2)$



$$TRAP(1) = \frac{4}{2}\left((-1)^4 + 3^4\right) = 2 \cdot 82 = 164$$

$$TRAP(2) = \frac{2}{2}\left((-1)^4 + 2(1)^4 + 3^4\right) = 84$$

$$E(2) \approx \frac{TRAP(2)-TRAP(1)}{3} = \frac{84-164}{3} = \frac{-80}{3}$$

$|E(2)| < .5$ No $\to$ Subdivide -

$-1 \quad 1 \qquad\qquad 1 \quad 3$

$E_1 = \frac{E_0}{2} = \frac{1}{4}$

$TRAP(1) = \frac{2}{2}((-1)^4 + 1^4) = 2$

$TRAP(2) = \frac{1}{2}((-1)^4 + 2 \cdot 0^4 + 1^4) = 1$

$E(2) \approx \frac{TRAP(2)-TRAP(1)}{3} = \frac{1-2}{3} = \frac{-1}{3}$

$|E(2)| = \frac{1}{3} > \frac{1}{4}$

probably just one more time

$E_1 = \frac{E_0}{2} = \frac{1}{4}$

$TRAP(1) = \frac{2}{2}(1^4 + 3^4) = 82$

$TRAP(2) = \frac{1}{2}(1^4 + 2 \cdot 2^4 + 3^4) = \frac{1}{2}(114)$ $1+32+81$ $= 57$

$E(2) = \frac{57-82}{2} = \frac{-25}{2}$

$|E(2)| = \frac{25}{2} > \frac{1}{4}$

probably several more times.